Transition Dynamics of VTOL Aircraft

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Asymptotic approximations using multiple time scales are developed for the longitudinal dynamics of VTOL vehicles during the hover, forward-flight transition. After reviewing the method briefly, the transitional equations of motion whose coefficients vary with flight velocity are studied. Rapid and slow motions of the vehicle are extracted individually and combined to yield a composite solution. The time scales are necessarily nonlinear functions and complex quantities to describe nonautonomous phenomena. Aircraft in steady flight arise as a natural limit. The present method thus generalizes earlier time scale analyses. The technique is applied to VTOL vehicles exhibiting typical stability derivative variations, first to the two degree-of-freedom case and then the three degree-of-freedom case. The asymptotic approximations are compared with exact numerical integrals. An analytical description of the aircraft motion is thus obtained. The problem of transition points is outlined.

1. Introduction

In the light of the recent advances in aviation technology, the operation of V/STOL vehicles is becoming more and more widespread. Consequently, there is an increasing need for a thorough understanding of the problems associated with the use of these vehicles. A central question is that of the stability and nature of the characteristic motion during the hover, forward flight transition. This is important from the standpoint of safety as well as performance. *Sui generis*, VTOL vehicles are difficult to study in regard to their dynamics. In contradistinction to the steady flight case, even the classical approach of linearizing the equations of motion about a reference flight condition is fraught with great difficulty, since the flight condition is itself changing. The system is of rather high order (third or fourth for motion in the plane of symmetry) and has nonconstant coefficients. The linearized equations are, therefore, analytically intractable in general. Exact analytical solutions cannot be obtained and hence approximations are resorted to. The importance of analytical solutions in providing insight about the complex physical system is self-evident. Curtiss[1] studied VTOL dynamics by an examination of the corresponding Riccati equation. He developed a modified root locus method to qualitatively predict the trends in the dynamics of the variable system with respect to a prescribed flight condition. However, this is valid only for short times and does not provide a uniform description of the motion. There is, therefore, a need for a uniformly valid analytical description of the aircraft motion. Other work in this area is discussed in Ref. 1.

In this paper we make use of the intrinsic characteristic time constants present in the dynamics and associate with them independent time scales on which the motion is observed. The fast and slow motions in the dynamics are separated in a systematic manner, and are combined to yield a composite description of the aircraft motion valid through the transition.

Presented as Paper 69-130 at the AIAA 7th Aerospace Sciences Meeting, New York, January 20–22, 1969; submitted January 15, 1969; revision received December 30, 1969. The author wishes to acknowledge many helpful and stimulating discussions with D. Graham, W. D. Hayes, and G. Sandri. The investigation was in part supported by the U.S. Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, under Contract AF 33(615)-3657, Project 8219, Task 821904 and was discussed in the author's doctoral dissertation from Princeton University (1967).

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2. Method of Multiple Scales

The multiple scales technique has its origin in the works of Krylov, Bogoliubov, and Mitropolsky[2] on nonlinear oscillations. By means of this technique, some nonlinear differential equations were studied by Cole and Kevorkian[3] and some problems in celestial mechanics by Kevorkian.[4] Frieman[5] and Sandri[6] have studied it in the context of irreversible statistical mechanics. Recently an application of the multiple scales technique to conventional aircraft dynamics was reported by H. Ashley.[7] He obtains approximations to the constant coefficient linear differential equations of aircraft motion using linear scales. However, it will be shown in this paper that for linear stationary systems, exact solutions can be obtained, instead of approximations, using only linear scales. All these applications employ linear time scales, i.e., the new time scales are linear functions of the original independent variable. The theory has been generalized by the author[8,11] to include nonlinear scales which may also be complex quantities. Indeed, for a large class of problems such as those describing nonlinear and nonautonomous phenomena, linear scales are inadequate and the more general concept of nonlinear scales is necessary. Furthermore, the time scales may be also considered as complex quantities.

The method of multiple scales was recently introduced as a mathematical technique to exploit the presence of a small (or large) parameter if one is available in a problem. The technique has an intuitive justification and can be seen to be physically meaningful. The mathematical theory as applied to linear time varying systems is developed by the author in Refs. 8 and 11. Application to some nonlinear nonautonomous systems has been discussed in Refs. 12 and 13. This paper primarily concerns an analysis of V/STOL vehicle motion through the hover-forward flight transition, by the technique of multiple scales. A brief discussion of the basic ideas and an account of the main mathematical results required for this study are given first.

2.1 Extension

The fundamental idea is to extend the domain of the independent variable into a space of higher dimension using suitable time scales or "clocks." These are determined by knowing the precise nature of the nonuniformities arising in direct perturbation theory. The ordinary differential equations are converted into a set of partial differential equations. These are solved such that, along certain lines in the extended space, their solutions become the same as those of the original equations. Such lines are termed "trajectories." It should be noted that the variables in general are not necessarily real. The clocks are so chosen that the new terms that arise due to extension, called "counter-terms," eliminate the nonuniformities of direct perturbation theory so that, in the extended domain, a uniform description of the unknown function may be obtained. The concept becomes more transparent by the study of a simple example.

Consider a physical phenomenon represented by a slowly decaying exponential,

$$f(t;\epsilon) = \exp(-\epsilon t); \quad 0 < \epsilon \ll 1 \tag{2.1}$$

The governing differential equation is

$$df/dt + \epsilon f = 0, \quad f(0) = 1 \tag{2.2}$$

A straight perturbation expansion would yield

$$f = 1 - \epsilon t + \epsilon^2 t^2/2! \ldots$$

which is nonuniform for $t \gtrsim 1/\epsilon$.

Physically we are trying to observe the phenomenon represented by f using a clock which beats on the t scale. An observer, therefore, will have to wait for a long time (the longer the smaller ϵ is) in order to notice a perceptible change in f. Another observer, who uses a "super" clock which measures t in giant units of t/ϵ, would see the phenomenon much better. For then, using the slow variable $\tau_1 = \epsilon t$, the phenomenon is described by $\mathbf{f}(\tau_1) = \exp(-\tau_1)$, which is a uniform description in τ_1.

Formally, we may write

$$\left.\begin{array}{c} t \rightarrow \{\tau_0, \tau_1\}; \quad \tau_0 = t, \ \tau_1 = \epsilon t \\ f(t) \rightarrow \mathbf{f}(\tau_0,\tau_1) \text{ and } (d/dt) \rightarrow (\partial/\partial\tau_0) + \epsilon(\partial/\partial\tau_1) \end{array}\right\} \tag{2.3}$$

Thus, using Eq. (2.2) and equating like powers of ϵ

$$\partial\mathbf{f}/\partial\tau_0 = 0 \quad \text{and} \quad \partial\mathbf{f}/\partial\tau_1 + \mathbf{f} = 0 \tag{2.4}$$

Therefore,

$$\mathbf{f}(\tau_0,\tau_1) = A(\tau_1) = \exp(-\tau_1) \tag{2.5}$$

$\mathbf{f}$ can be represented in a three-dimensional space (Fig. 1) with orthogonal axes τ_0, τ_1, and $\mathbf{f}$. Readings on the fast and slow scales are represented by points along τ_0 and τ_1, respectively. The function is a constant along τ_0 but decays exponentially along τ_1. These two behaviors have been extracted separately. The method of multiple scales thus enables us to perform readings on appropriate scales by employing a sufficient number of independent observers.

2.2 Nonlinear Scales

In dealing with equations with variable coefficients the simple choice of linear scales is not adequate. This can be illustrated by the first-order equation[8]

$$df/dt + \epsilon\omega(t)f = 0 \tag{2.6}$$

The choice of Eq. (2.3) leads to the set of equations

$$\left.\begin{array}{c} \partial\mathbf{f}/\partial\tau_0 = 0 \\ \partial\mathbf{f}/\partial\tau_1 + \omega(\tau_0)\mathbf{f} = 0 \end{array}\right\} \tag{2.7}$$

Integrating: $\mathbf{f}(\tau_0,\tau_1) = A(\tau_1) = c\exp[-\omega(\tau_0)\tau_1]$

We notice a contradiction unless ω is a constant. On the other hand, if we choose, $\tau_0 = t; \tau_1 = \epsilon k(t)$ (k is a "clock" function, to be determined), Eq. (2.7) becomes

$$\dot{k}(\tau_0)\partial\mathbf{f}/\partial\tau_1 + \omega(\tau_0)\mathbf{f} = 0 \tag{2.8}$$

i.e.,

$$(A'/A)(\tau_1) = -(\omega/\dot{k})(\tau_0) = \text{const} \tag{2.9}$$

Therefore,

$$\mathbf{f} = C\exp(-\tau_1), \ \tau_1 = \epsilon\int\omega dt \tag{2.10}$$

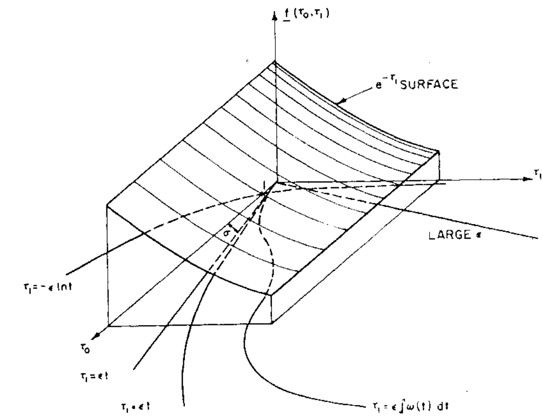

Fig. 1 Function surface in extended space (line under f means boldface).

yielding the exact solution. Thus, a nonlinear time scale is absolutely essential even in the case of a first-order equation with a variable coefficient. With reference to Fig. 1 it can be seen that the trajectory of $\tau_1(\tau_0)$ is nonlinear and the projection of the extended function surface along the trajectory still gives a uniform description of the solution. Indeed, the exact solution is obtained.

But second- and higher-order equations with variable coefficients do not, in general, yield exact solutions and hence we look for approximations. For second-order equations with slowly varying coefficients a well-known approximation is the Liouville-Green or WKBJ approximation. In Refs. 8 and 11 this has been derived by the multiple scales technique. Further, the theory had been generalized to linear differential equations of order n, with slowly varying coefficients. We shall quote the main result below and then apply it to VTOL dynamics.

2.3 Multiple Scales Formula for *n*th Order Equation[8]

Consider the equation

$$y^{(n)} + \lambda\omega_{n-1}(t)y^{(n-1)} + \ldots + \lambda^{n-1}\omega_1(t)y^{(1)} + \lambda^n\omega_0(t)y = 0 \tag{2.11}$$

where $|\lambda| \gg 1$. The asymptotic approximation as $\lambda \rightarrow \infty$ is given by[8]

$$\bar{y}(\tau_0,\tau_1) = \sum_{i=1}^{n} c_i \left(\frac{\partial F}{\partial \dot{k}}(\tau_0)\right)^{-1/2} \gamma(\tau_0)\exp(\tau_{1i}) \tag{2.12}$$

where

$$\frac{d}{d\tau_0}(\ln\gamma) = \frac{\partial}{\partial\tau_0}\left\{\ln\left(\frac{\partial F}{\partial\dot{k}}\right)^{1/2}\right\} \qquad F(\dot{k},\tau_0) \equiv (\dot{k})^n + \omega_{n-1}(\tau_0)(\dot{k})^{n-1} + \ldots + \omega_1(\tau_0)\dot{k} + \omega_0(\tau_0) \tag{2.13}$$

Under certain conditions γ can be considered a constant.[8] $k(\tau_0)$ is determined from the characteristic equation

$$F(\dot{k},\tau_0) = 0 \tag{2.14}$$

$\tau_0 = t, \ \tau_1 = \lambda k(t)$ is the extension and c_i are arbitrary constants. The roots of the characteristic equation are assumed to be distinct for this analysis.

An equation with slowly varying coefficients can be written in the form Eq. (2.11). For example, the equation

$$y^{(n)} + \omega_{n-1}(\epsilon t)y^{(n-1)} + \ldots + \omega_1(\epsilon t)y^{(1)} + \omega_0(\epsilon t)y = 0 \tag{2.15}$$

(where $\epsilon \ll 1$) can be written as in Eq. (2.11) by using the transformation $\tau = \epsilon t$ and $\lambda = 1/\epsilon$.

It can be easily seen that Eq. (2.12) yields the Liouville-Green or WKBJ solution for second-order equation. τ_0 and τ_1 are identified as slow and fast time scales, as $\lambda \gg 1$.

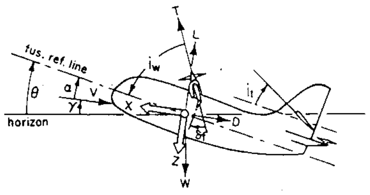

Fig. 2 Axis system and notation.

3. Aircraft Equations of Motion

The motion of the aircraft is considered with reference to a system of body axes fixed in the vehicle (Fig. 2). Only rectilinear motion in the vehicle's plane of symmetry is considered and the effects of elastic deformation are assumed to be negligible. Under the usual assumptions the longitudinal equations of motion can be written in conventional notation[9] as

$$u' + wq + g\sin\theta = X(u,w,q,\delta_T,\delta_E,i_w) \quad (3.1a)$$

$$w' - uq - g\cos\theta = Z(u,w,q,\delta_T,\delta_E,i_w) \quad (3.1b)$$

$$\dot{q} = M(u,w,q,\delta_T,\delta_E,i_w) \quad (3.1c)$$

$$\dot{\theta} = q \quad (3.1d)$$

These equations are nonlinear and nonautonomous in general. In a tilt-wing vehicle, δ_T, δ_E, and i_w represent the control parameters, denoting propeller blade pitch, pitching moment control, and wing-tilt angle, respectively. Instead of dealing with the complete nonlinear and nonautonomous equations, they are simplified in order to allow an analytical treatment and enable qualitative conclusions to be drawn. The equations of motion are linearized in the usual way by making the following assumptions. The motion is considered about a steady level flight and the vehicle is fully trimmed, i.e., in a state of equilibrium with all the forces of moments balanced out. If the vehicle now encounters a disturbance such that the resulting motion is small in magnitude, the motion is described by a set of linear differential equations with constant coefficients. The homogeneous perturbation equations are given by

$$\begin{aligned} u' - X_u u - X_w w + g\theta &= 0 \\ w' - Z_w w - Z_u u - V\dot{\theta} &= 0 \\ \theta'' - M_q\dot{\theta} - M_u u - M\cdot\dot{w} &= 0 \end{aligned} \quad (3.2)$$

For a conventional airplane at cruising flight the stability derivatives are constants. The perturbed transient motion can be determined by solving the coupled equations with constant coefficients. For a VTOL vehicle executing a transition, the flight condition varies from instant to instant; hence the aerodynamic parameters of the vehicle, since they depend on the flight condition, also vary through the transition. The vehicle is still assumed to be trimmed throughout. Control required to trim is not considered in this analysis, and we consider transitions at level flight. Nonlevel flight transitions can also be studied without much difficulty. The coefficients of the linearized equations are, therefore, treated as variable if the time history of the trim conditions can be predicted. Further, this change in the coefficient is assumed to arise primarily from the change in flight velocity, although, in general, they depend on the wing tilt angle i_w and the power setting δ_T.

Qualitatively, the following observations can be made. At forward flight a VTOL vehicle behaves essentially like a conventional airplane and at hover like a helicopter, in regard to dynamic motion. The forces and moments produced by the propeller and the wing-slipstream interaction largely influence the low-speed characteristics of a VTOL vehicle. Near cruising speeds these effects become less important. The stability derivatives have constant values corresponding to hover and forward flight, but change continuously from one set to the other as the vehicle accelerates until it attains cruising velocity.

At first glance we may just consider the dynamics of the "frozen system" corresponding to hover and cruising flight. This assumes that the coefficients are essentially constants. At hover the characteristic roots of the constant coefficient system consist of a complex conjugate pair with positive real part and a pair of negative real roots. The motion, therefore, exhibits oscillatory instability. In cruising flight the motion is characterized by two pairs of complex conjugate roots, usually with negative real parts, corresponding to the conventional airplane. The transition is, therefore, from a helicopter-like vehicle to an airplane-like one with the accompanying difficulties in the analysis and control of the vehicle.

4. Application to Aircraft Dynamics

We shall now consider the application of the theory of multiple scales to study the dynamics of flight vehicles. First we shall consider conventional aircraft and then go on to VTOL vehicles.

4.1 Conventional Aircraft

Ashley[7] considers the constant coefficient equations describing a conventional aircraft in the light of multiple scales and obtains approximate solutions, order by order. He also achieves a rough separation of the performance and the dynamic response problems. For both these questions, he employs linear time scales in the fashion of Kevorkian.[4] We note, however, that using linear scales, it is possible to recover exact solutions of linear equations with constant coefficients. This is done by choosing a proper function of the small parameter as the expansion parameter. For example, in solving the first order equation,

$$(1 - \epsilon)y' + y = 0 \quad (4.1)$$

the choice of $\tau_n = \epsilon^n t$ leads to an infinite number of scales. However, the choice of $\delta = 1/(1 - \epsilon)$ as the expansion parameter results in complete solution with only two scales. An example which is typical of conventional aircraft, is used for illustration.

Consider the equation

$$y^{(4)} + 2(a + \epsilon)y^{(3)} + (b + 4a\epsilon + 2\epsilon^2)y^{(2)} + 2\epsilon(b + 2a\epsilon)y^{(1)} + 2b\epsilon^2 y = 0 \quad (4.2)$$

where a,b are constants of order unity and $0 < \epsilon \ll 1$. Direct perturbation leads to secular terms. In order to uniformize the solution, let us choose the simple extension Eq. (2.3). Subscripts 1 and 2 denote differentiation w.r.t. τ_0 and τ_1, respectively. The lowest-order extended equation is given by

$$y_{1111} + 2ay_{111} + by_{11} = 0 \quad (4.3)$$

Integration leads to

$$y(\tau_0,\tau_1) = A(\tau_1)e^{m_1\tau_0} + B(\tau_1)e^{m_2\tau_0} + C(\tau_1)\tau_0 + D(\tau_1) \quad (4.4)$$

where m_1,m_2 satisfy

$$m^2 + 2am + b = 0 \quad (4.5)$$

Higher-order equations are given by

$$2y_{1112} + 3ay_{112} + y_{111} + by_{12} + 2ay_{11} + by_1 = 0 \quad (4.6)$$

$$6y_{1122} + 6ay_{122} + 6y_{112} + by_{22} + 8ay_{12} + 2y_{11} + 2by_1 + 4ay_1 + 2by = 0 \quad (4.7)$$

Substituting Eq. (4.4) in (4.6) and noting the linear independence of the exponentials, A and B are deduced to be pure constants and $C = E\exp(-\tau_1)$. Substituting this in

Eq. (4.7) and simplifying, we obtain,

$$E = 0, D = \rho_1 \exp(n_1\tau_1) + \rho_2 \exp(n_2\tau_1) \tag{4.8}$$

where ρ_1,ρ_2 are constants and n_1n_2 satisfy the equation

$$n^2 + 2n + 2 = 0 \tag{4.9}$$

The solution of Eq. (4.2) is, therefore, given by,

$$y(\tau_0,\tau_1) = c_1 \exp(m_1\tau_0) + c_2 \exp(m_2\tau_0) + c_3 \exp(n_1\tau_1) + c_4 \exp(n_2\tau_1)$$

where m_1,m_2 and n_1,n_2 satisfy Eq. (4.5) and (4.9), respectively. The restriction $\tau_0 = t$, $\tau_1 = \epsilon t$ leads to the exact solution of Eq. (4.2). This example can be considered to represent the motion of a conventional aircraft in the longitudinal mode; m_1,m_2 and n_1,n_2 describe the short period mode and the phugoid mode, respectively.

We thus see that by a proper choice of the separation parameter ϵ, exact solutions can be obtained using linear time scales. A simple example was chosen to illustrate this point. Since the solution of linear equations with constant coefficients is made up of exponentials of an argument linear in the independent variable, the only difference comes about in the linearity constant. Clearly a proper choice of extension will simplify the analysis and derive exact solutions.

4.2 VTOL Aircraft

We shall now study the motion during transition of VTOL aircraft in the light of multiple time scales. The transition is assumed to progress slowly so that the stability derivatives are slowly varying functions of the flight velocity. It must be emphasized that the "frozen system" approximation, often used in engineering analysis of slowly varying systems, must be used carefully and mainly for qualitative purposes. Otherwise, even slowly varying systems may lead to erroneous conclusions. We shall illustrate this point by constructing a simple model of this genre, to represent the VTOL vehicle in regard to stability of the solution during a hover-forward transition.

Simple dynamic model

Consider the system described by the equation,

$$y' - [(1 - \epsilon t)/(1 + \epsilon t)]y = 0 \tag{4.10}$$

where $0 < \epsilon \ll 1$, and $y(0) = 1$.

The characteristic root is at +1 at $t = 0$ and changes continuously as t increases and approaches -1 as $t \to \infty$. The frozen system indicates instability at first and then stability as $t \to \infty$. The simple frozen approximation is a growing exponential and does not match the true solution anywhere except at $t = 0$ and gives incorrect stability information. Another approximation, which is a slightly more refined scheme of "freezing" the system, is to treat the coefficient essentially as a constant as far as the solution is concerned, but to vary on a slower ϵt time scale. This can be viewed as a simple application of the time scales method. The approximation $\bar{y}(t;\epsilon) = \exp\{(1 - \epsilon t)/(1 + \epsilon t)\}t$ thus obtained gives the correct initial behavior and stability information, but is quite wrong in representing the true solution in other respects.

Table 1 Stability derivative variation

Stability derivative	Functional form
$-X_u$	0.2
$-Z_w$	0.1 + 0.004V
$-Z_u$	0.25V/(10 + V)
$-M_q$	0.1 + 0.0034V
$-M_u$	0.015(−1 + V/150)
$-M_w$	(i) (−0.02 + 0.00025)V/150
	(ii) 0.005 + 0.015(V/150)^2
Two degree-of-freedom case:	$V(t) = 150t/(10 + t)$ fps
Three degree-of-freedom case:	$V(t) = 150t/(20 + t)$ fps

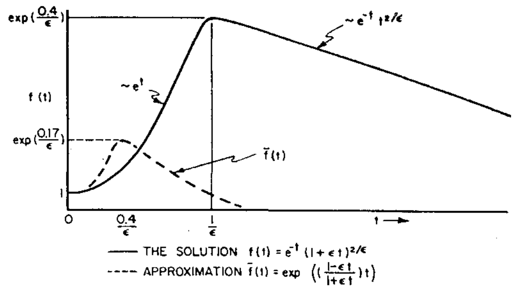

Fig. 3 Simple dynamic model (not to scale).

The multiple scales method, being general, offers unlimited freedom in the choice of the scales. The choice of a proper extension leads to a uniformly valid solution. Let us choose the extension

$$t \to \{\tau_0,\tau_1\}$$
$$\tau_0 = 1;\ \text{i.e., } \tau_0 = t + \text{const} \tag{4.11}$$
$$\tau_1 = (1/\epsilon)k(t)$$

Equation (4.10) can be written as

$$(1/\epsilon + t)dy/dt - (1/\epsilon - t)y = 0$$

This suggests that the constant in τ_0 is $0(1/\epsilon)$. Let $\tau_0 = t + c/\epsilon$ where $c = 0(1)$. The extended equations are

$$(1 - c)\dot{k}(\partial y/\partial \tau_1) = 0 \tag{4.12a}$$

$$(1 - c)(\partial y/\partial \tau_0) - (1 + c)y + \tau_0\dot{k}(\partial y/\partial \tau_1) = 0 \tag{4.12b}$$

$$\tau_0(\partial y/\partial \tau_0 + y) = 0 \tag{4.12c}$$

The choice of $c = 1$ from Eq. (4.12a) and $y = A(\tau_1)\exp(-\tau_0)$ from Eq. (4.12c) yields in (4.12b),

$$(A'/A)(\tau_1) = (2/\tau_0\dot{k})(\tau_0) = c_1 = \text{const} \tag{4.13}$$

whence

$$A = D \exp(c_1\tau_1), \quad k = 2/c_1 \ln\tau_0$$

Restricting along the trajectories defined by extension, we obtain

$$y(t) = \exp[-t + (2/\epsilon)\ln(1 + \epsilon t)] = e^{-t}(1 + \epsilon t)^{2/\epsilon} \tag{4.14}$$

which is the exact solution. It is seen that the generality of a nonlinear clock function is mandatory; the clock itself can be a highly nonlinear function even in simple problems. The solutions are sketched in Fig. 3.

4.3 Tilt Wing Vehicle

We shall now consider a specific example, a tilt-wing VTOL vehicle. Reference 10 contains a comparative study of the longitudinal stability derivatives of some typical tilt-wing VTOL vehicles. The present analysis employs the stability derivative variations as proposed in Ref. 10. The functional dependence of the coefficients in Eq. (3.2) on velocity is given in Table 1. The wing-tilt angle i_w is in control of the pilot so that any variation of $i_w(t)$ through the transition can be programed. The dependence of trim velocity V on wing angle is assumed to be linear and hence $V(t)$ can be chosen conveniently. The stability derivatives are now expressed as functions of t and this leads to a set of time-varying, coupled linear differential equations with slowly varying coefficients. The derivatives X_w and $M_{\dot{w}}$ are neglected since their contribution to the vehicle dynamics is considered to be small.

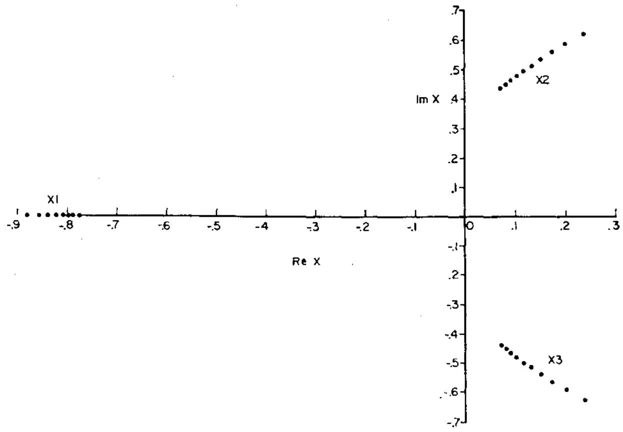

Fig. 4 Characteristic roots; third-order u equation.

Two degree-of-freedom case

Near hover, the two degree-of-freedom approximation is employed, in which the vertical or plunging motion is suppressed. At hover, the damped vertical mode is completely decoupled and has little effect on the other two modes. The system is represented by the following equations:

$$\begin{aligned} u' - X_u u + g\theta &= 0 \\ w' - Z_w w &= 0 \\ \theta'' - M_q\theta' - M_u u &= 0 \end{aligned} \tag{4.15}$$

The w mode is completely decoupled leaving the u and θ equations still coupled. On decoupling these by cross-differentiation we obtain,

$$u''' - (X_u + M_q)u'' + (X_u M_q - 2X'_u)u' + (gM_u + X'_u M_q - X''_u)u = 0 \tag{4.16}$$

$$\theta''' - \left(X_u + M_q + \frac{M'_u}{M_u}\right)\theta'' + \left(X_u M_q - M'_q + M_q\frac{M'_u}{M_u}\right)\theta' + gM_u\theta = 0 \tag{4.17}$$

On substituting for the coefficients from Table 1, these become

$$(1 + 0.1t)u''' + (0.3 + 0.081t)u'' + (0.02 + 0.0122t)u' + 0.48u = 0 \tag{4.18}$$

$$(1 + 0.1t)^2\theta''' + (0.4 + 0.081t)(1 + 0.1t)\theta'' + (0.081 + 0.0183t + 0.00122t^2)\theta' + 0.48(1 + 0.1t)\theta = 0 \tag{4.19}$$

$$(1 + 0.1t)w' + (0.1 + 0.07t)w = 0 \tag{4.20}$$

Equation (4.20) can be readily integrated to give:

$$w(t) = c\exp\left[-\int\left(\frac{0.1 + 0.07t}{1 + 0.1t}\right)dt\right] \tag{4.21}$$

Equations (4.16) and (4.17) are much more difficult to solve and in general, cannot be solved exactly. Equations (4.18) and (4.19) are solved approximately using the theory of time scales as embodied in Sec. 2.3. For example, the characteristic for Eq. (4.18) is given by

$$(1 + 0.1t)\dot{k}^3 + (0.3 + 0.081t)\dot{k}^2 + (0.02 + 0.0122t)\dot{k} + 0.48 = 0 \tag{4.22}$$

The roots are computed and plotted from $t = 0$ as $t \to \infty$. The root locus is given in Fig. 4. The linear combination of the independent asymptotic approximations as well as the exact numerical integrals were computed for various initial conditions. Some of these are depicted in Figs. 5–8. It is seen that the approximations represent the exact solutions very well uniformly through the transition. We can thus retain to some extent the familiar ideas of the analysis of linear stationary systems. The root loci for the u and θ equations are seen to be different. This is because of the new terms (which depend on the time derivatives of the stability derivatives) introduced by decoupling the coupled equations. A constant coefficient analysis of the decoupled equations yields the "frozen" approximation. It is seen that this does not represent the true solution anywhere after the first cycle. On the other hand, in the light of multiple scales, the fast time scale τ_1 shows up as a quadrature over the root variation and describes the frequency of the rapidly varying motion.

The slow amplitude variation (i.e., the envelope of the oscillatory solutions) is picked up on the slow time scale τ_0. A combination of the two results in a uniform description of the phenomenon in both amplitude and frequency.

Three degree-of-freedom case

This case, decoupling the longitudinal equations is an involved task. The stability derivatives X_w and M_w are still considered relatively less important and are neglected. Again, unlike the stationary case, the equations describing each dependent variable will not all be the same so that the time histories of u,w, and θ will be different. After a rather cumbersome decoupling process, the equation for u is given as

$$\begin{aligned} & u'''' - \left(X_u + Z_w + M_q + \frac{M'_w}{M_w}\right)u''' + \left(X_u Z_w + Z_w M_q + M_q X_u - VM_w - M'_q - 3X'_u + \frac{M'_w}{M_w}(X_u + M_q)\right)u'' + \\ & \left[-X_u Z_w M_q + VM_w X_u + gM_u - \frac{M'_w}{M_w}X_u M_q - X_u M'_q + 2X'_u\left(Z_w + M_q + \frac{M'_w}{M_w}\right) - 3X_u''\right]u' + \left[g\left(Z_u M_w - Z_w M_u - \frac{M'_w}{M_w}M_u - M'_u\right) - X_u'\left(Z_w M_q - VM_w - M'_q + \frac{M'_w}{M_w}M_q\right) + X''_u\left(Z_w + M_q + \frac{M'_w}{M_w}\right)\right]u = 0 \end{aligned} \tag{4.23}$$

When M_w is set to zero this equation reduces to Eq. (4.16), the two-degree-of-freedom case, valid at hover (X_u is a constant). However, we can check that when all coefficients

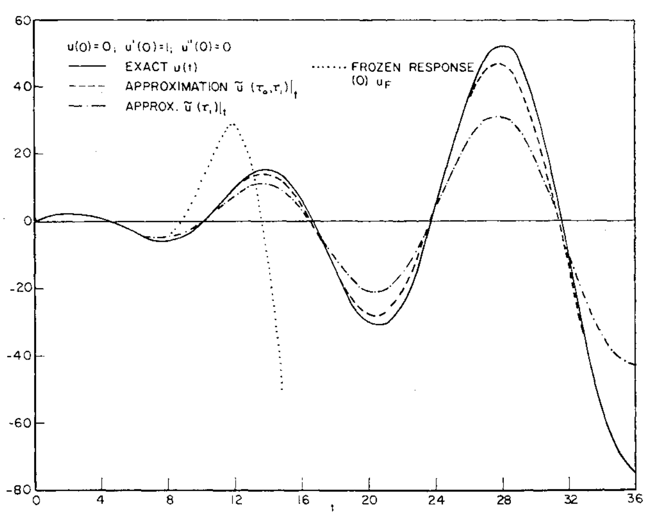

Fig. 5 Solutions, $u(t)$, $u_{\text{frozen}}(t)$, $\tilde{u}(\tau_1)|_t$, $\tilde{u}(\tau_0,\tau_1)|_t$.

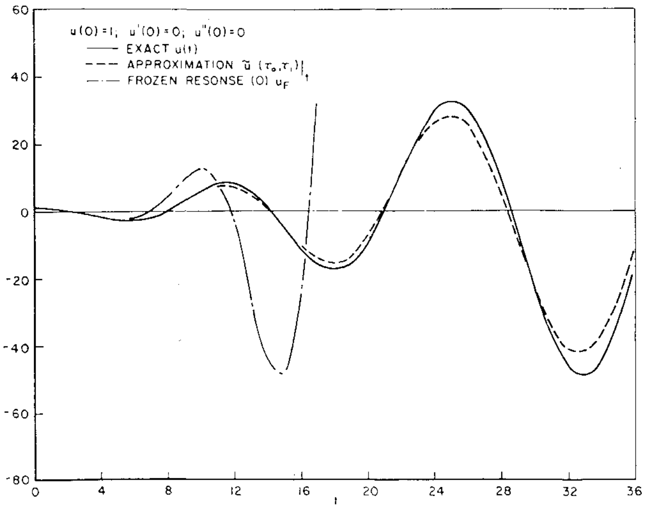

Fig. 6 Solutions; $u(t), u_{\text{frozen}}(t), \tilde{u}(\tau_0,\tau_1)|_t$.

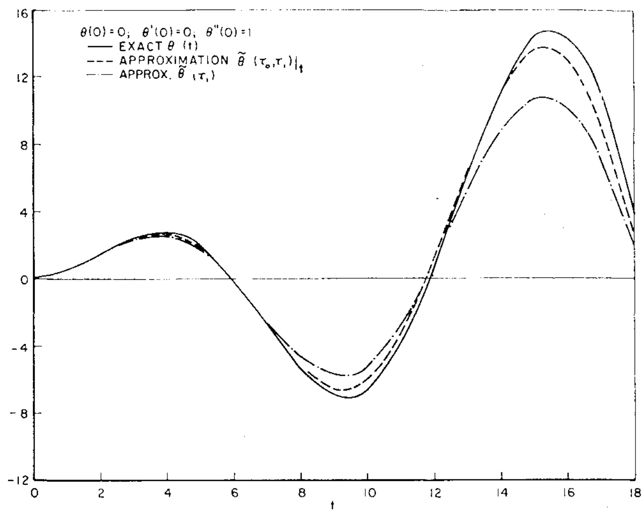

Fig. 8 Solutions; $\theta(t)$, $\tilde{\theta}(\tau_1)$,$\tilde{\theta}(\tau_0,\tau_1)|_t$.

are constants the equation becomes

$$u'''' - (X_u + Z_w + M_q)u''' + (X_uZ_w + Z_wM_q + M_qX_u - VM_w)u'' - (X_uZ_wM_q - VM_wX_u + gM_u)u' + g(Z_uM_w - Z_wM_u)u = 0 \quad (4.24)$$

Now substitution of the assumed stability derivative variation from Table 1, leads to the following equation for u:

$$p_4u^{(4)} + p_3u^{(3)} + p_2u^{(2)} + p_1u^{(1)} + p_0u = 0 \quad (4.25)$$

where

$$p_4 = (10 + 16t)(10 + t)^3(-0.2 + 0.0175t)t$$

$$p_3 = (2 - 1.35t - 0.232t^2 + 0.02643t^3)(10 + 16t)(10 + t)^2$$

$$p_2 = (60 + 13.7t - 10.4875t^2 + 4.19t^3 - 0.9988t^4 + 0.058t^5)(10 + 16t)$$

$$p_1 = (4 - 100.78t - 12.523t^2 + 1.6994t^3 - 0.123t^4 + 0.0107t^5)(10 + 16t)$$

$$p_0 = 3.22(10 + t)(30.3 + 23.07t - 15.7t^2 - 1.4573t^3 + 0.0034t^4 + 0.0095t^5) \quad (4.26)$$

The equation has regular singular points at $t = 0$ and $t \approx 11.42$, corresponding to the zeros of $M_w(t)$. Roughly speaking, near $t = 0$ the fourth-order equation is approximated by a third-order equation which has only two degrees of freedom. The other singular point occurs in a region in which two of the characteristic roots coalesce and the time scales approximation fails in this region. This is the transition point problem and will be discussed later.

Different stability derivative variations were studied. Approximations were derived for various initial conditions and were found to represent the exact solutions well in each case. Figures 9 and 10 illustrate the root locus and the response for a typical case. It is seen that the frozen approximation breaks down after the first cycle while the time scales solution describes the aircraft motion well, but with an error in phase in the latter half of the transition. A uniform description of the aircraft motion is thus obtained. The approximate solutions to the homogeneous equations of motion constitute the approximate variable weighting function (or impulse response) for the time varying system. This is given by,

$$\tilde{g}(t,t_i) = \sum_{i=1}^{n} c_i \left(\frac{\partial F}{\partial \dot{k}}(t)\right)^{-1/2} \exp\left(\lambda \int_{t_1}^{t} \dot{k}_i(s)ds\right) \quad (4.27)$$

where t_i = time of impulse; t = time of observation, and F, k, λ are as defined in Sec. 2.3. c_i are constants chosen according to initial conditions. The Fourier transform of this would lead to an approximation for Zadeh's system function. The approximate variable impulse response is depicted in Fig. 11.

Two of the characteristic roots coalesce on the real axis before breaking away into the complex plane. The asymptotic approximation fails for a condition of multiple roots. This constitutes the transition point (or turning point) problem. On one side of the transition point, the roots are real and yield nonoscillatory solutions. On the other side the roots are complex and yield oscillatory solutions. The transition is, therefore, from nonoscillatory to oscillatory

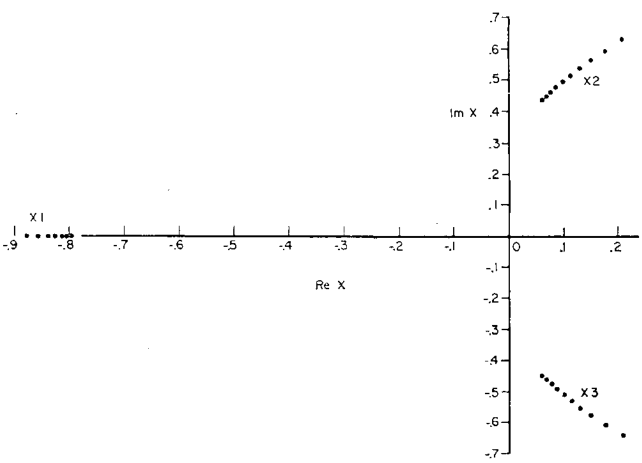

Fig. 7 Characteristic roots; third-order θ equation.

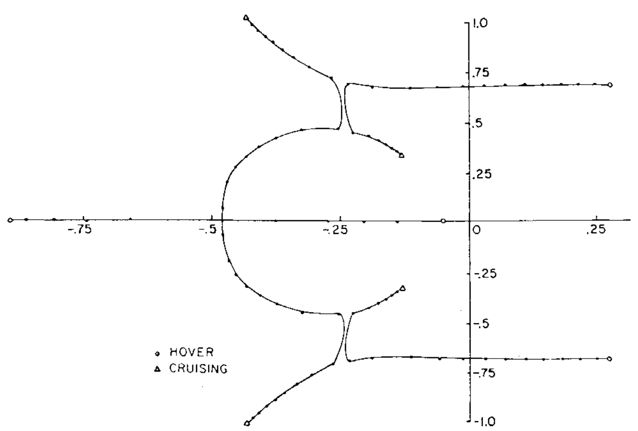

Fig. 9 Characteristic roots; fourth-order u equation; case 4.

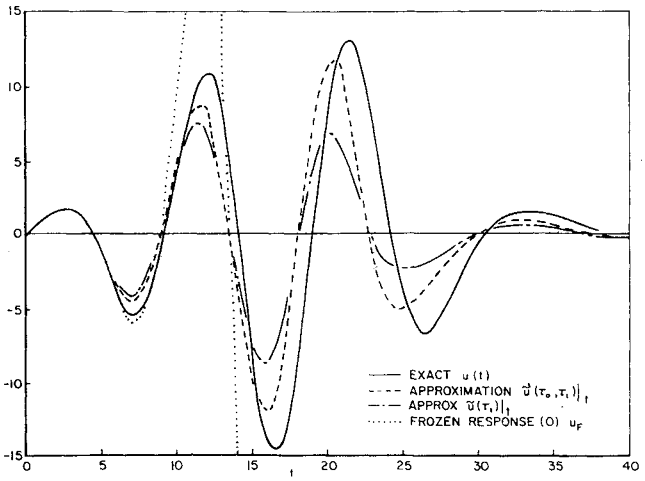

Fig. 10 Case 4; (0, 1, 0, 0) IC. Solutions; $u(t)$ $u_{\text{frozen}}(t)$, $\tilde{u}(\tau_1)_t, \tilde{u}(\tau_0,\tau_1)|_t$.

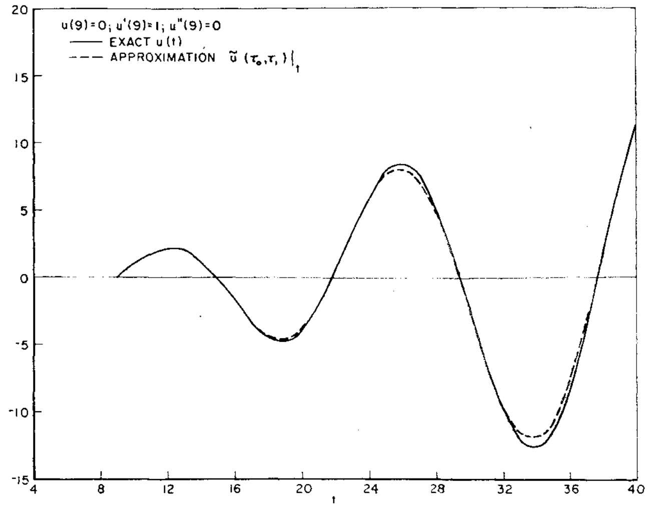

Fig. 11 Variable impulse response.

behavior. In general, the exact solution cannot be represented in terms of elementary functions throughout the domain of interest. However, such a representation is possible in restricted regions. In particular, when the coefficients of a linear differential equation are slowly varying the solutions can be represented by elementary functions in an asymptotic sense. Such a representation breaks down when two or more roots coalesce because the approximate solution then becomes unbounded. In this region nothing simpler than a nonelementary function will suffice. Therefore, higher transcendental functions such as hypergeometric functions could be used in such a region, to represent the phenomenon. Alternatively, the asymptotic solutions valid on either side of a transition point could be joined in an intricate but specifiable manner. This would lead to the connection formulae.

For the VTOL problem, usually one transition point on the real axis is encountered. A suitable choice of initial conditions will isolate the phugoid mode which will be free from transition points. On the other hand, the use of a digital computer for the approximation precludes any difficulty with the transition point, and the solution progresses through the troublesome region with mainly an error in phase. And while studying the natural dynamics of the vehicle, great accuracy in the phase of the response is not very important, since it merely represents the orientation of the vehicle in space. The rates of change of frequency and amplitude are usually of interest and this is described very well by the approximations.

5. Concluding Remarks

It is seen that the author's generalization of the multiple scales technique has provided an analytical solution that represents the aircraft motion to good accuracy.

The frozen approximation is good only for very short times, because the error becomes large in less than a cycle of the oscillation. The rapid time scale approximation represents the frequency of the solution well. The amplitude variation is obtained by the slower scale and a combination of the two yields the complete behavior. A uniform description of the aircraft motion has been obtained in terms of the coefficients. The effect of the stability derivatives on the motion of the vehicle can, therefore, be studied analytically. Further, it is felt that this paper established a basis for the variable characteristic roots concept in the analysis of time varying systems using multiple scales.

In obtaining the approximations, only the independent variable was extended into a space of two dimensions. This corresponds to the lowest order in an expansion of the dependent variable. In obtaining higher-order approximations we have freedom—to extend the independent variable into higher dimensions, and to extend the dependent variable, or a combination of both. This offers unlimited choices which may be utilized to improve the approximations, if greater numerical accuracy is desired.

For STOL vehicles the variations in the aerodynamic parameters are not as rapid as for VTOL aircraft. However, the effects are qualitatively similar. The technique presented in this paper can be readily applied to the study of STOL aircraft dynamics as well. Other flight dynamic problems of a nonautonomous nature, such as the rocket flight (variable mass) through a variable density atmosphere, etc., can be studied in a similar manner.

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JULY 1970 AIAA JOURNAL VOL. 8, NO. 7

Optimal Two- and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit

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Minimum-fuel, multiple-impulse orbital rendezvous is investigated for the case in which the transfer time is specified (time-fixed case). A method for obtaining optimal solutions is employed which is applicable to rendezvous or orbit transfer between elliptical orbits of low eccentricity. In this method optimal solutions are constructed by satisfying the necessary conditions for the primer vector. It is assumed that the terminal orbits lie close enough to an intermediate circular reference orbit that the linearized equations of motion can be used to describe the transfer. The linear boundary value problem for the impulse magnitudes for rendezvous is then solved analytically. As an application of the method, optimal two- and three-impulse fixed-time rendezvous transfers between coplanar circular orbits are obtained for a range of transfer times. These linearized solutions combined with previously obtained four-impulse solutions provide a complete solution for fixed-time coplanar circle-to-circle rendezvous between close orbits for transfer times up to nearly two terminal orbit periods.

Nomenclature

$(\dot{\ })$ = first derivative of ( ) with respect to time. Time derivatives of vectors are taken with respect to an inertial reference frame
$(\ )'$ = first derivative with respect to dimensionless time τ
$|(\ )|$ = determinant of the array ( )
a = radius of reference circular orbit
$A\#, B\#, C\#, D\#$ = arbitrary constants in Sec. V
B_{Fj} = partition of state transition matrix (Eq. 38)
b = variable defined by Eq. (31)
c = $\cos\theta_2$
$\mathbf{h}_j$ = vector column of H matrix
H = matrix defined by Eq. (3)
I = identity matrix
k = variable defined by Eq. (11)
k_j = variable defined by Eq. (33)
m = variable defined by Eq. (10)
m_j = variable defined by Eq. (32)
N_{2j} = partition of transition matrix [Eq. (40)]
$\mathbf{P}$ = primer vector
q_j = variable defined by Eq. (12)
$\mathbf{r}$ = position vector
δR = nondimensional difference between final and initial circular orbit radii
s = $\sin\theta_2$
t = time
T_{Fj} = partition of transition matrix [Eq. (40)]
$\mathbf{u}_j$ = unit vector in direction of jth thrust impulse
$\mathbf{v}$ = velocity vector
$\Delta\mathbf{V}_j$ = vector velocity change due to jth thrust impulse
$\Delta\mathbf{V}$ = vector defined by Eq. (1)
$\mathbf{w}_j$ = vector column of W matrix
W = matrix defined by Eq. (1)
$\mathbf{x}$ = position-velocity state vector
β = initial phase angle of target (Fig. 12)
$\delta(\ )$ = first variation of ( )
$\Delta(\ )$ = change in ( )
θ = central angle
θ_2 = $\tau_2 - \tau_1$ (Sec. V)
λ = radial component of primer vector
μ = circumferential component of primer vector
$\rho(\ )$ = rank of ( )
τ = dimensionless time (see Appendix)
Φ_{ji} = state transition matrix between times t_i and t_j
ω = mean motion of reference orbit

Subscripts

0 = initial value
F = final value
H = half-transfer time [Eq. (3)]
j = time of jth impulse
r = radial component
θ = circumferential component

Superscript

$+$ = initial coast if precedes variable; final coast if succeeds variable

I. Introduction

IN the study of minimum fuel trajectories in an inverse square gravitational field, considerable attention has been directed to an aspect of the problem which has been called

Received April 14, 1969; revision received December 22, 1969. This paper is based upon part of the author's Sc. D. dissertation in the Massachusetts Institute of Technology (M.I.T.) Department of Aeronautics and Astronautics. The author wishes to acknowledge the many helpful comments of T. N. Edelbaum. The research was performed at the M.I.T. Experimental Astronomy Laboratory under NASA Grant NsG 254-62.

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